5.7 Additional Techniques of Integration

1. $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx = \int (1-\cos^2 x)\cos^2 x \sin x\,dx$

$$= \int (1-u^2)u^2(-du) \qquad [u=\cos x,\ du=-\sin x\,dx]$$

$$= \int (u^2-1)u^2\,du = \int (u^4-u^2)\,du = \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C$$

2. $\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x\,dx$

$$= \int_0^1 (1-u^2)^2\,du \qquad [u=\sin x,\ du=\cos x\,dx]$$

$$= \int_0^1 (1-2u^2+u^4)\,du = \left[u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5\right]_0^1 = \left(1-\tfrac{2}{3}+\tfrac{1}{5}\right) - 0 = \tfrac{8}{15}$$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,(1-\sin^2 x)\cos x\,dx$

$$= \int_1^{\sqrt{2}/2} u^5(1-u^2)\,du \qquad [u=\sin x,\ du=\cos x\,dx]$$

$$= \int_1^{\sqrt{2}/2} (u^5-u^7)\,du = \left[\tfrac{1}{6}u^6 - \tfrac{1}{8}u^8\right]_1^{\sqrt{2}/2} = \left(\tfrac{1/8}{6} - \tfrac{1/16}{8}\right) - \left(\tfrac{1}{6}-\tfrac{1}{8}\right) = -\tfrac{11}{384}$$

4. $\int \sin^3(mx)\,dx = \int (1-\cos^2 mx)\sin mx\,dx = -\frac{1}{m}\int (1-u^2)\,du \qquad [u=\cos mx,\ du=-m\sin mx\,dx]$

$$= -\tfrac{1}{m}\left(u - \tfrac{1}{3}u^3\right) + C = -\tfrac{1}{m}\left(\cos mx - \tfrac{1}{3}\cos^3 mx\right) + C = \tfrac{1}{3m}\cos^3 mx - \tfrac{1}{m}\cos mx + C$$

5. $\int_0^{2\pi} \cos^2(6\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi} [1+\cos(12\theta)]\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{12}\sin(12\theta)\right]_0^{2\pi} = \frac{1}{2}[(2\pi+0)-(0+0)] = \pi$

6. $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x \cos^2 x)\,dx = \int_0^{\pi/2} \frac{1}{4}(2\sin x \cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx$

$$= \tfrac{1}{4}\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 4x)\,dx = \tfrac{1}{8}\int_0^{\pi/2} (1-\cos 4x)\,dx = \tfrac{1}{8}\left[x - \tfrac{1}{4}\sin 4x\right]_0^{\pi/2} = \tfrac{1}{8}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{16}$$

7. Let $u=\sec x$. Then $du = \sec x \tan x\,dx$, so

$$\int \tan^3 x \sec x\,dx = \int (\tan^2 x)(\tan x \sec x)\,dx = \int (\sec^2 x - 1)(\sec x \tan x\,dx)$$

$$= \int (u^2-1)\,du = \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}\sec^3 x - \sec x + C$$

8. Let $u=\sec x$, so $du = \sec x \tan x\,dx$. Thus,

$$\int \tan^5 x \sec^3 x\,dx = \int \tan^4 x \sec^2 x\,(\sec x \tan x)\,dx = \int (\sec^2 x - 1)^2 \sec^2 x\,(\sec x \tan x\,dx)$$

$$= \int (u^2-1)^2 u^2\,du = \int (u^6 - 2u^4 + u^2)\,du$$

$$= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C$$

9. Let $u=\tan x$. Then $du = \sec^2 x\,dx$, so

$$\int_0^{\pi/4} \tan^2 x \sec^4 x\,dx = \int_0^{\pi/4} \tan^2 x \sec^2 x\,(\sec^2 x\,dx) = \int_0^{\pi/4} \tan^2 x\,(1+\tan^2 x)(\sec^2 x\,dx)$$

$$= \int_0^1 u^2(1+u^2)\,du = \int_0^1 (u^2+u^4)\,du = \left[\tfrac{1}{3}u^3 + \tfrac{1}{5}u^5\right]_0^1 = \tfrac{1}{3} + \tfrac{1}{5} = \tfrac{8}{15}$$

10. Let $u=\tan x$. Then $du = \sec^2 x\,dx$, so

$$\int \tan^4 x \sec^6 x\,dx = \int \tan^4 x \sec^4 x\,(\sec^2 x\,dx) = \int \tan^4 x(1+\tan^2 x)^2\,(\sec^2 x\,dx)$$

$$= \int u^4(1+u^2)^2\,du = \int (u^8 + 2u^6 + u^4)\,du$$

$$= \tfrac{1}{9}u^9 + \tfrac{2}{7}u^7 + \tfrac{1}{5}u^5 + C = \tfrac{1}{9}\tan^9 x + \tfrac{2}{7}\tan^7 x + \tfrac{1}{5}\tan^5 x + C$$

11. $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\,d\theta$ and

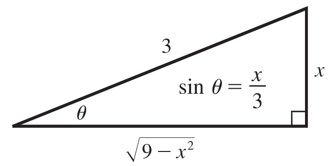
$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\,|\cos\theta| = 3\cos\theta$. (Note that $\cos\theta \ge 0$ because $-\pi/2 \le \theta \le \pi/2$.)

Thus, substitution gives

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C$$

Since this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express $\cot\theta$ in terms of $\sin\theta = x/3$ or by drawing a diagram, as shown, where θ is interpreted as an angle of a right triangle.



Since $\sin\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9-x^2}$, so we can simply read the value of $\cot\theta$ from the figure: $\cot\theta = \dfrac{\sqrt{9-x^2}}{x}$.

(Although $\theta > 0$ in the diagram, this expression for $\cot\theta$ is valid even when $\theta < 0$.) Since $\sin\theta = x/3$, we have

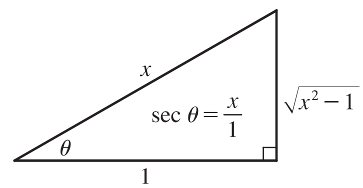
$\theta = \sin^{-1}(x/3)$ and so $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$

12. $x = \sec\theta$, where $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$. Then

$dx = \sec\theta\,\tan\theta\,d\theta$ and

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ (since

$\tan\theta \ge 0$ for the specified values of θ). Thus, substitution gives



$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\sec\theta\,\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec^3\theta}d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^3\theta\,d\theta$$

$$= \int \sin^2\theta\,\cos\theta\,d\theta = \int u^2 du \quad [u = \sin\theta,\, du = \cos\theta\,d\theta]$$

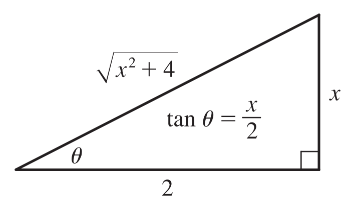
$$= \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C = \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = \frac{(x^2-1)^{3/2}}{3x^3} + C$$

13. $x = 2\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+4} = \sqrt{(2\tan\theta)^2 + 4} = \sqrt{4\tan^2\theta + 4}$$

$$= \sqrt{4(\tan^2\theta + 1)} = 2\sqrt{\sec^2\theta} = 2\,|\sec\theta|$$

$$= 2\sec\theta \quad [\text{since } \sec\theta \ge 0 \text{ for } -\pi/2 < \theta < \pi/2].$$



Thus, substitution gives

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\,(2\sec\theta)}\left(2\sec^2\theta\,d\theta\right) = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{u^2}\,du \qquad [u = \sin\theta,\, du = \cos\theta\,d\theta]$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4}\,\frac{1}{\sin\theta} + C = -\frac{1}{4}\cdot\frac{\sqrt{x^2+4}}{x} + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

14. (a) $\dfrac{d}{d\theta}\left[\frac{1}{2}(\sec\theta\,\tan\theta+\ln|\sec\theta+\tan\theta|)+C\right]$

$$=\frac{1}{2}\left(\sec\theta\cdot\sec^2\theta+\tan\theta\cdot\sec\theta\,\tan\theta+\frac{1}{\sec\theta+\tan\theta}\cdot\sec\theta\,\tan\theta+\sec^2\theta\right)+0$$

$$=\frac{1}{2}\left[\sec\theta\left(\sec^2\theta+\tan^2\theta\right)+\frac{\sec\theta\,(\tan\theta+\sec\theta)}{\sec\theta+\tan\theta}\right]$$

$$=\tfrac{1}{2}\left[\sec\theta\left(\sec^2\theta+\sec^2\theta-1\right)+\sec\theta\right]=\tfrac{1}{2}\sec\theta\left[\left(2\sec^2\theta-1\right)+1\right]$$

$$=\tfrac{1}{2}\sec\theta\left(2\sec^2\theta\right)=\sec^3\theta.$$

Thus, $\int\sec^3\theta\,d\theta=\frac{1}{2}(\sec\theta\,\tan\theta+\ln|\sec\theta+\tan\theta|)+C.$

(b) As in Exercise 13, we use the substitution $x=\tan\theta$, where $-\pi/2<\theta<\pi/2$. Then $dx=\sec^2\theta\,d\theta$ and $\sqrt{x^2+1}=\sqrt{\tan^2\theta+1}=\sqrt{\sec^2\theta}=|\sec\theta|=\sec\theta$ (since $\sec\theta\geq 0$ for $-\pi/2<\theta<\pi/2$). When $x=0$, $\tan\theta=0 \;\Rightarrow\; \theta=0$, and when $x=1$, $\tan\theta=1 \;\Rightarrow\; \theta=\frac{\pi}{4}$. Thus, substitution gives

$$\int_0^1\sqrt{x^2+1}\,dx=\int_0^{\pi/4}\sec\theta\left(\sec^2\theta\,d\theta\right)=\int_0^{\pi/4}\sec^3\theta\,d\theta=\left[\tfrac{1}{2}(\sec\theta\,\tan\theta+\ln|\sec\theta+\tan\theta|)\right]_0^{\pi/4}$$

$$=\tfrac{1}{2}\left[\left(\sqrt{2}\cdot 1+\ln\left|\sqrt{2}+1\right|\right)-\left(1\cdot 0+\ln\left|1+0\right|\right)\right]$$

$$=\tfrac{1}{2}\left[\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right]$$

15. Let $t=\sec\theta$, so $dt=\sec\theta\,\tan\theta\,d\theta$, $t=\sqrt{2} \;\Rightarrow\; \theta=\frac{\pi}{4}$, and $t=2 \;\Rightarrow\; \theta=\frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2}\frac{1}{t^3\sqrt{t^2-1}}\,dt=\int_{\pi/4}^{\pi/3}\frac{1}{\sec^3\theta\,\tan\theta}\sec\theta\,\tan\theta\,d\theta=\int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta=\int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$

$$=\int_{\pi/4}^{\pi/3}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta=\tfrac{1}{2}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$=\tfrac{1}{2}\left[\left(\tfrac{\pi}{3}+\tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right)-\left(\tfrac{\pi}{4}+\tfrac{1}{2}\cdot 1\right)\right]=\tfrac{1}{2}\left(\tfrac{\pi}{12}+\tfrac{\sqrt{3}}{4}-\tfrac{1}{2}\right)=\tfrac{\pi}{24}+\tfrac{\sqrt{3}}{8}-\tfrac{1}{4}$$

16. Let $x=4\sin\theta$, where $-\pi/2\leq\theta\leq\pi/2$. Then $dx=4\cos\theta\,d\theta$ and $\sqrt{16-x^2}=\sqrt{16-16\sin^2\theta}=\sqrt{16\cos^2\theta}=4\,|\cos\theta|=4\cos\theta$. When $x=0$, $4\sin\theta=0 \;\Rightarrow\; \theta=0$, and when $x=2\sqrt{3}$, $4\sin\theta=2\sqrt{3} \;\Rightarrow\; \sin\theta=\frac{\sqrt{3}}{2} \;\Rightarrow\; \theta=\frac{\pi}{3}$. Thus, substitution gives

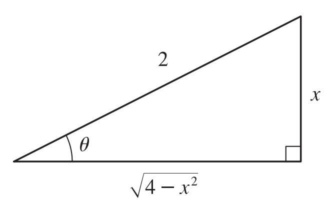
$$\int_0^{2\sqrt{3}}\frac{x^3}{\sqrt{16-x^2}}\,dx=\int_0^{\pi/3}\frac{4^3\sin^3\theta}{4\cos\theta}\,4\cos\theta\,d\theta=4^3\int_0^{\pi/3}\sin^3\theta\,d\theta$$

$$=4^3\int_0^{\pi/3}(1-\cos^2\theta)\,\sin\theta\,d\theta=-4^3\int_1^{1/2}(1-u^2)\,du\quad[u=\cos x,\,du=-\sin x\,dx]$$

$$=-64\left[u-\tfrac{1}{3}u^3\right]_1^{1/2}=-64\left[\left(\tfrac{1}{2}-\tfrac{1}{24}\right)-\left(1-\tfrac{1}{3}\right)\right]=-64\left(-\tfrac{5}{24}\right)=\tfrac{40}{3}$$

Or: Let $u=16-x^2$, $x^2=16-u$, $du=-2x\,dx$.

17. Let $x=2\sin\theta$, where $-\pi/2\leq\theta\leq\pi/2$. Then $dx=2\cos\theta\,d\theta$ and $\sqrt{4-x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2\,|\cos\theta|=2\cos\theta$.



Thus, $\displaystyle\int\frac{dx}{x^2\sqrt{4-x^2}}=\int\frac{2\cos\theta}{4\sin^2\theta(2\cos\theta)}\,d\theta=\frac{1}{4}\int\csc^2\theta\,d\theta$

$$=-\frac{1}{4}\cot\theta+C=-\frac{\sqrt{4-x^2}}{4x}+C\quad\text{[see figure]}$$

18. Let $x = \tan\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = \sec^2\theta\, d\theta$ and $\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$.

Then $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta = \int \tan^3\theta\sec\theta\,d\theta = I$. Now let $u = \sec\theta$, so $du = \sec\theta\tan\theta\,d\theta$ and

$$\begin{aligned} I &= \textstyle\int \tan^2\theta\,(\sec\theta\tan\theta\,d\theta) = \int(\sec^2\theta - 1)\,(\sec\theta\tan\theta\,d\theta) = \int(u^2-1)\,du \\ &= \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}\sec^3\theta - \sec\theta + C = \tfrac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C \end{aligned}$$

19. (a) $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

20. (a) $\dfrac{x}{x^2+x-2} = \dfrac{x}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$

(b) $\dfrac{x^2}{x^2+x+2} = \dfrac{(x^2+x+2)-(x+2)}{x^2+x+2} = 1 - \dfrac{x+2}{x^2+x+2}$

Notice that x^2+x+2 can't be factored because its discriminant is $b^2 - 4ac = -7 < 0$.

21. $\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(2x+1)(x-1)$ to get $5x+1 = A(x-1) + B(2x+1) \;\Rightarrow$

$5x+1 = Ax - A + 2Bx + B \;\Rightarrow\; 5x+1 = (A+2B)x + (-A+B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A + 2B = 5$ and

$-A + B = 1$. Adding these equations gives us $3B = 6 \;\Leftrightarrow\; B = 2$, and hence, $A = 1$. Thus,

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \left(\frac{1}{2x+1} + \frac{2}{x-1}\right)dx = \tfrac{1}{2}\ln|2x+1| + 2\ln|x-1| + C.$$

Another method: Substituting 1 for x in the equation $5x+1 = A(x-1) + B(2x+1)$ gives $6 = 3B \;\Leftrightarrow\; B = 2$.

Substituting $-\frac{1}{2}$ for x gives $-\frac{3}{2} = -\frac{3}{2}A \;\Leftrightarrow\; A = 1$.

22. $\dfrac{x-4}{x^2-5x+6} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$. Multiply both sides by $(x-2)(x-3)$ to get $x-4 = A(x-3) + B(x-2) \;\Rightarrow$

$x - 4 = Ax - 3A + Bx - 2B \;\Rightarrow\; x-4 = (A+B)x + (-3A-2B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A + B = 1$ and $-3A - 2B = -4$.

Adding twice the first equation to the second gives us $-A = -2 \;\Leftrightarrow\; A = 2$, and hence, $B = -1$.Thus,

$$\begin{aligned} \int_0^1 \frac{x-4}{x^2-5x+6}\,dx &= \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right)dx = \Big[2\ln|x-2| - \ln|x-3|\Big]_0^1 \\ &= (0 - \ln 2) - (2\ln 2 - \ln 3) = -3\ln 2 + \ln 3 \;\;[\text{or } \ln\tfrac{3}{8}] \end{aligned}$$

Another method: Substituting 3 for x in the equation $x - 4 = A(x-3) + B(x-2)$ gives $-1 = B$. Substituting 2 for x

gives $-2 = -A \;\Leftrightarrow\; A = 2$.

23. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1 = A(x-1) + B(x+1) \;\Rightarrow$

$1 = Ax - A + Bx + B \;\Rightarrow\; 1 = (A+B)x + (-A+B)$. The coefficients of x must be equal and the constant terms are also equal, so $A+B=0$ and $-A+B=1$. Adding these equations gives us $2B=1 \quad\Leftrightarrow\quad B=\frac{1}{2}$, and hence, $A=-\frac{1}{2}$. Thus

$$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1|+\tfrac{1}{2}\ln|x-1|\right]_2^3$$
$$= \left(-\tfrac{1}{2}\ln 4+\tfrac{1}{2}\ln 2\right)-\left(-\tfrac{1}{2}\ln 3+\tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2+\ln 3-\ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]$$

Another method: Substituting 1 for x in the equation $1=A(x-1)+B(x+1)$ gives $1=2B \quad\Leftrightarrow\quad B=\frac{1}{2}$.

Substituting -1 for x gives $1=-2A \quad\Leftrightarrow\quad A=-\frac{1}{2}$.

24. $\dfrac{x^2+2x-1}{x^3-x}=\dfrac{x^2+2x-1}{x(x+1)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get

$x^2+2x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1) \quad\Rightarrow$

$x^2+2x-1=Ax^2-A+Bx^2-Bx+Cx^2+Cx \quad\Rightarrow$

$x^2+2x-1=(A+B+C)x^2+(-B+C)x-A$. Equating constant terms, we get $-A=-1 \quad\Leftrightarrow\quad A=1$.

Equating coefficients of x^2 gives $1=1+B+C \quad\Leftrightarrow\quad 0=B+C$. Equating coefficients of x gives $2=-B+C$.

Adding these equations gives $2=2C \quad\Leftrightarrow\quad C=1$, and hence, $B=-1$. Thus,

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int\left(\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x-1}\right)dx = \ln|x|-\ln|x+1|+\ln|x-1|+C = \ln\left|\frac{x(x-1)}{x+1}\right|+C.$$

Another method: Substituting 0 for x in the equation $x^2+2x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$ gives $-1=-A \quad\Leftrightarrow\quad A=1$. Substituting -1 for x gives $-2=2B \quad\Leftrightarrow\quad B=-1$. Substituting 1 for x gives $2=2C \quad\Leftrightarrow\quad C=1$.

25. $\dfrac{10}{(x-1)(x^2+9)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)\left(x^2+9\right)$ to get

$10=A\left(x^2+9\right)+(Bx+C)(x-1)$ $(\star)$. Substituting 1 for x gives $10=10A \quad\Leftrightarrow\quad A=1$. Substituting 0 for x gives $10=9A-C \quad\Rightarrow\quad C=9(1)-10=-1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0=A+B \quad\Rightarrow\quad B=-1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx = \int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx$$
$$= \ln|x-1|-\tfrac{1}{2}\ln(x^2+9)-\tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right)+C$$

In the second term we used the substitution $u=x^2+9$ and in the last term we used Formula 10.

26. $\dfrac{2x^2+5}{(x^2+1)(x^2+4)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+4}$. Multiply both sides by $\left(x^2+1\right)\left(x^2+4\right)$ to get

$2x^2+5=(Ax+B)\left(x^2+4\right)+(Cx+D)\left(x^2+1\right) \quad\Leftrightarrow$

$2x^2+5=\left(Ax^3+Bx^2+4Ax+4B\right)+\left(Cx^3+Dx^2+Cx+D\right) \quad\Leftrightarrow$

$2x^2+5=(A+C)x^3+(B+D)x^2+(4A+C)x+(4B+D)$. Comparing coefficients gives us $A+C=0$, $B+D=2$, $4A+C=0$, and $4B+D=5$. Solving gives us $A=C=0$ and $B=D=1$. Thus,

$$\int \frac{2x^2+5}{(x^2+1)(x^2+4)}\,dx = \int\left(\frac{1}{x^2+1}+\frac{1}{x^2+4}\right)dx = \tan^{-1}x+\tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right)+C.$$

27. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$x^3+x^2+2x+1 = (Ax+B)(x^2+2)+(Cx+D)(x^2+1) \quad \Leftrightarrow$

$x^3+x^2+2x+1 = (Ax^3+Bx^2+2Ax+2B)+(Cx^3+Dx^2+Cx+D) \quad \Leftrightarrow$

$x^3+x^2+2x+1 = (A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$. Comparing coefficients gives us the following system of equations:

$$A+C=1 \quad \textbf{(1)} \qquad B+D=1 \quad \textbf{(2)}$$
$$2A+C=2 \quad \textbf{(3)} \qquad 2B+D=1 \quad \textbf{(4)}$$

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=0$. Subtracting equation **(2)** from equation **(4)** gives us $B=0$, so $D=1$. Thus, $I = \displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int\left(\frac{x}{x^2+1}+\frac{1}{x^2+2}\right)dx$. For $\displaystyle\int \frac{x}{x^2+1}\,dx$, let $u=x^2+1$ so $du=2x\,dx$ and then $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln(x^2+1)+C$. For $\displaystyle\int\frac{1}{x^2+2}\,dx$, use Formula 10 with $a=\sqrt{2}$. So $\displaystyle\int\frac{1}{x^2+2}\,dx = \int\frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$.

Thus, $I = \dfrac{1}{2}\ln(x^2+1)+\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}+C$.

28. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x}+\dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2-x+6 = A(x^2+3)+(Bx+C)x$. Substituting 0 for x gives $6=3A \quad\Leftrightarrow\quad A=2$. The coefficients of the x^2-terms must be equal, so $1=A+B \quad\Rightarrow$ $B=1-2=-1$. The coefficients of the x-terms must be equal, so $-1=C$. Thus,

$$\int\frac{x^2-x+6}{x^3+3x}\,dx = \int\left(\frac{2}{x}+\frac{-x-1}{x^2+3}\right)dx = \int\left(\frac{2}{x}-\frac{x}{x^2+3}-\frac{1}{x^2+3}\right)dx$$
$$= 2\ln|x|-\frac{1}{2}\ln(x^2+3)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$$

29. $\displaystyle\int\frac{x}{x-6}\,dx = \int\frac{(x-6)+6}{x-6}\,dx = \int\left(1+\frac{6}{x-6}\right)dx = x+6\ln|x-6|+C$

30. $\displaystyle\int\frac{r^2}{r+4}\,dr = \int\left(\frac{r^2-16}{r+4}+\frac{16}{r+4}\right)dr = \int\left(r-4+\frac{16}{r+4}\right)dr$ [or use long division]

$= \frac{1}{2}r^2-4r+16\ln|r+4|+C$

31.

$$\begin{array}{r|l} & \quad\quad\quad\quad x \\ \hline x^2+4 & x^3 + 0x^2 + 0x + 4 \\ & x^3 \quad\quad\quad + 4x \\ \hline & \quad\quad -4x + 4 \end{array}$$

By long division, $\dfrac{x^3+4}{x^2+4} = x+\dfrac{-4x+4}{x^2+4}$. Thus,

$$\int\frac{x^3+4}{x^2+4}\,dx = \int\left(x+\frac{-4x+4}{x^2+4}\right)dx = \int\left(x-\frac{4x}{x^2+4}+\frac{4}{x^2+2^2}\right)dx$$
$$= \frac{1}{2}x^2-4\cdot\frac{1}{2}\ln|x^2+4|+4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C = \frac{1}{2}x^2-2\ln(x^2+4)+2\tan^{-1}\left(\frac{x}{2}\right)+C$$

32. $\dfrac{x^3-4x-10}{x^2-x-6}=x+1+\dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)}=\dfrac{A}{x-3}+\dfrac{B}{x+2}$. Then $3x-4=A(x+2)+B(x-3)$. Taking $x=3$ and $x=-2$, we get $5=5A \;\Leftrightarrow\; A=1$ and $-10=-5B \;\Leftrightarrow\; B=2$, so

$$\begin{aligned}\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx &= \int_0^1\left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)dx=\left[\frac{1}{2}x^2+x+\ln|x-3|+2\ln(x+2)\right]_0^1\\ &=\left(\tfrac12+1+\ln 2+2\ln 3\right)-(0+0+\ln 3+2\ln 2)=\tfrac32+\ln 3-\ln 2=\tfrac32+\ln\tfrac32\end{aligned}$$

33. Let $u=\sqrt{x}$, so $u^2=x$ and $dx=2u\,du$. Thus,

$$\begin{aligned}\int_9^{16}\frac{\sqrt{x}}{x-4}\,dx &= \int_3^4 \frac{u}{u^2-4}2u\,du=2\int_3^4\frac{u^2}{u^2-4}\,du=2\int_3^4\left(1+\frac{4}{u^2-4}\right)du \qquad \text{[by long division]}\\ &=2+8\int_3^4\frac{du}{(u+2)(u-2)}\quad (\star)\end{aligned}$$

Multiply $\dfrac{1}{(u+2)(u-2)}=\dfrac{A}{u+2}+\dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1=A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac14$ and $A=-\frac14$, so $\dfrac{1}{(u+2)(u-2)}=\dfrac{-1/4}{u+2}+\dfrac{1/4}{u-2}$ and $(\star)$ is

$$\begin{aligned}2+8\int_3^4\left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right)du &= 2+8\left[-\tfrac14\ln|u+2|+\tfrac14\ln|u-2|\right]_3^4=2+\Big[2\ln|u-2|-2\ln|u+2|\Big]_3^4\\ &=2+2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4=2+2\left(\ln\tfrac26-\ln\tfrac15\right)=2+2\ln\tfrac{2/6}{1/5}\\ &=2+2\ln\tfrac53 \;\text{ or }\; 2+\ln\left(\tfrac53\right)^2=2+\ln\tfrac{25}{9}\end{aligned}$$

34. Let $u=\sqrt{x+3}$, so $u^2=x+3$ and $2u\,du=dx$. Then

$\displaystyle\int\frac{dx}{2\sqrt{x+3}+x}=\int\frac{2u\,du}{2u+(u^2-3)}=\int\frac{2u}{u^2+2u-3}\,du=\int\frac{2u}{(u+3)(u-1)}\,du$. Now

$\dfrac{2u}{(u+3)(u-1)}=\dfrac{A}{u+3}+\dfrac{B}{u-1} \quad\Rightarrow\quad 2u=A(u-1)+B(u+3)$. Setting $u=1$ gives $2=4B$, so $B=\frac12$.

Setting $u=-3$ gives $-6=-4A$, so $A=\frac32$. Thus,

$$\begin{aligned}\int\frac{2u}{(u+3)(u-1)}\,du &= \int\left(\frac{\frac32}{u+3}+\frac{\frac12}{u-1}\,du\right)\\ &=\tfrac32\ln|u+3|+\tfrac12\ln|u-1|+C=\tfrac32\ln\left(\sqrt{x+3}+3\right)+\tfrac12\ln\left|\sqrt{x+3}-1\right|+C\end{aligned}$$

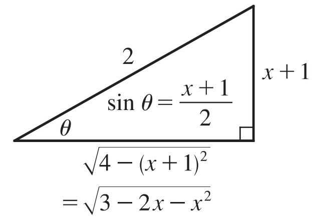
35. $x^2+x+1=x^2+x+\frac14+1-\frac14$ [add and subtract the square of one-half the coefficient of x to complete the square]

$$=x^2+x+\tfrac14+\tfrac34=\left(x+\tfrac12\right)^2+\left(\tfrac{\sqrt3}{2}\right)^2$$

So $I=\displaystyle\int\frac{dx}{x^2+x+1}=\int\frac{1}{\left(x+\frac12\right)^2+\left(\frac{\sqrt3}{2}\right)^2}\,dx$. Now let $u=x+\frac12 \quad\Rightarrow\quad du=dx$ and

$$I=\int\frac{1}{u^2+\left(\frac{\sqrt3}{2}\right)^2}\,du=\frac{1}{\frac{\sqrt3}{2}}\tan^{-1}\frac{u}{\frac{\sqrt3}{2}}+C=\frac{2}{\sqrt3}\tan^{-1}\frac{2\left(x+\frac12\right)}{\sqrt3}+C=\frac{2}{\sqrt3}\tan^{-1}\frac{2x+1}{\sqrt3}+C.$$

36. $3 - 2x - x^2 = 3 - \left(x^2 + 2x\right) = 3 - \left(x^2 + 2x + 1 - 1\right) = 4 - (x+1)^2$.



$$I = \int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{x}{\sqrt{4 - (x+1)^2}}\,dx.$$

Let $x + 1 = 2\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$, so $dx = 2\cos\theta\,d\theta$ and

$$\sqrt{4 - (x+1)^2} = \sqrt{4 - 4\sin^2\theta} = 2\sqrt{\cos^2\theta} = 2\,|\cos\theta| = 2\cos\theta.$$

Thus,

$$\begin{aligned} I &= \int \frac{2\sin\theta - 1}{2\cos\theta}(2\cos\theta\,d\theta) = \int (2\sin\theta - 1)\,d\theta \\ &= -2\cos\theta - \theta + C = -2\,\frac{\sqrt{3 - 2x - x^2}}{2} - \sin^{-1}\frac{x+1}{2} + C \\ &= -\sqrt{3 - 2x - x^2} - \sin^{-1}\frac{x+1}{2} + C \end{aligned}$$

5.8 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. Let $u = \pi x$, so that $du = \pi\,dx$. Then

$$\begin{aligned} \int \tan^3(\pi x)\,dx &= \int \tan^3 u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}\int \tan^3 u\,du \overset{69}{=} \tfrac{1}{\pi}\left[\tfrac{1}{2}\tan^2 u + \ln|\cos u|\right] + C \\ &= \tfrac{1}{2\pi}\tan^2(\pi x) + \tfrac{1}{\pi}\ln|\cos(\pi x)| + C \end{aligned}$$

2. $\displaystyle\int e^{2\theta}\sin 3\theta\,d\theta \overset{98}{=} \frac{e^{2\theta}}{2^2 + 3^2}(2\sin 3\theta - 3\cos 3\theta) + C = \tfrac{2}{13}e^{2\theta}\sin 3\theta - \tfrac{3}{13}e^{2\theta}\cos 3\theta + C$

3. Let $u = 2x$ and $a = 3$. Then $du = 2\,dx$ and

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{4x^2+9}} &= \int \frac{\frac{1}{2}\,du}{\frac{u^2}{4}\sqrt{u^2+a^2}} = 2\int \frac{du}{u^2\sqrt{a^2+u^2}} \overset{28}{=} -2\,\frac{\sqrt{a^2+u^2}}{a^2u} + C \\ &= -2\,\frac{\sqrt{4x^2+9}}{9\cdot 2x} + C = -\frac{\sqrt{4x^2+9}}{9x} + C \end{aligned}$$

4. $\displaystyle\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx = \int_4^6 \frac{1}{\left(\frac{1}{2}u\right)^2\sqrt{u^2-7}}\left(\frac{1}{2}\,du\right) \qquad [u = 2x,\ du = 2\,dx]$

$$= 2\int_4^6 \frac{du}{u^2\sqrt{u^2-7}} \overset{45}{=} 2\left[\frac{\sqrt{u^2-7}}{7u}\right]_4^6 = 2\left(\frac{\sqrt{29}}{42} - \frac{3}{28}\right) = \frac{\sqrt{29}}{21} - \frac{3}{14}$$

5. Let $u = e^x$, so that $du = e^x\,dx$ and $e^{2x} = u^2$. Then

$$\begin{aligned} \int e^{2x}\arctan(e^x)\,dx &= \int u^2\arctan u\left(\frac{du}{u}\right) = \int u\arctan u\,du \\ &\overset{92}{=} \frac{u^2+1}{2}\arctan u - \frac{u}{2} + C = \frac{1}{2}(e^{2x}+1)\arctan(e^x) - \frac{1}{2}e^x + C \end{aligned}$$

$dv = e^x\,dx \quad\Rightarrow\quad du = (k+1)x^k\,dx,\ v = e^x$, we get

$$\int x^{k+1}e^x\,dx = x^{k+1}e^x - (k+1)\int x^k e^x\,dx = x^{k+1}e^x - (k+1)\left[e^x \sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}x^i\right]$$

$$= e^x\left[x^{k+1} - (k+1)\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}x^i\right] = e^x\left[x^{k+1} + \sum_{i=0}^{k}(-1)^{k-i+1}\frac{(k+1)k!}{i!}x^i\right]$$

$$= e^x\sum_{i=0}^{k+1}(-1)^{(k+1)-i}\frac{(k+1)!}{i!}x^i$$

This verifies S_n for $n = k+1$. Thus, by mathematical induction, S_n is true for all n, where n is a positive integer.

5.9 Approximate Integration

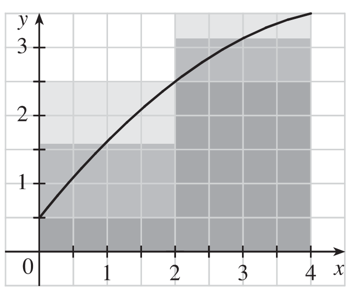
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$$

(b)



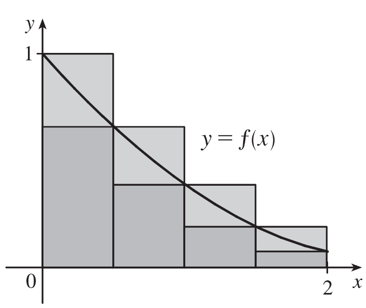
L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 39 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$

This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 39 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

2.



The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that $L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

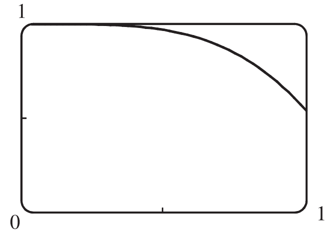
(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

3. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

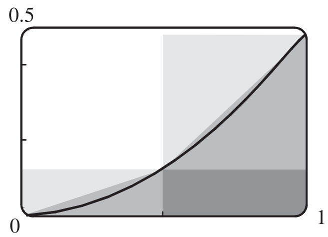
(a) $T_4 = \frac{1}{4 \cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0, 1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos(x^2)\,dx < 0.908907$.

4.



(a) Since f is increasing on $[0, 1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I. Since f is concave upward on $[0, 1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n, we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5 = \left(\frac{1}{2}\,\Delta x\right)[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

5. (a) $f(x) = \dfrac{x}{1+x^2}$, $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{10} = \dfrac{1}{5}$

$M_{10} = \frac{1}{5}\left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{5}{10}\right) + \cdots + f\left(\frac{19}{10}\right)\right] \approx 0.806598$

(b) $S_{10} = \frac{1}{5 \cdot 3}\left[f(0) + 4f\left(\frac{1}{5}\right) + 2f\left(\frac{2}{5}\right) + 4f\left(\frac{3}{5}\right) + 2f\left(\frac{4}{5}\right) + \cdots + 4f\left(\frac{9}{5}\right) + f(2)\right] \approx 0.804779$

Actual: $I = \displaystyle\int_0^2 \frac{x}{1+x^2}\,dx = \left[\tfrac{1}{2}\ln\left|1+x^2\right|\right]_0^2 \qquad [u = 1 + x^2,\ du = 2x\,dx]$

$= \frac{1}{2}\ln 5 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 5 \approx 0.804719$

Errors: $E_M = \text{actual} - M_{10} = I - M_{10} \approx -0.001879$

$E_S = \text{actual} - S_{10} = I - S_{10} \approx -0.000060$

6. (a) $f(x) = x\cos x$, $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$

$M_4 = \frac{\pi}{4}\left[f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right)\right] \approx -1.945744$

(b) $S_4 = \frac{\pi}{4 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{2\pi}{4}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right] \approx -1.985611$

Actual: $I = \int_0^\pi x\cos x\,dx = \Big[x\sin x + \cos x\Big]_0^\pi \qquad$ [use parts with $u = x$ and $dv = \cos x\,dx$]

$= (0 + (-1)) - (0 + 1) = -2$

Errors: $E_M = \text{actual} - M_4 = I - M_4 \approx -0.054256$

$E_S = \text{actual} - S_4 = I - S_4 \approx -0.014389$

7. $f(x) = \sqrt[4]{1+x^2}$, $\Delta x = \dfrac{2-0}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{1}{4 \cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + \cdots + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.413790$

(b) $M_8 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 2.411453$

(c) $S_8 = \frac{1}{4 \cdot 3}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + 2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.412232$

8. $f(x) = \sin\left(x^2\right)$, $\Delta x = \dfrac{\frac{1}{2}-0}{4} = \dfrac{1}{8}$

(a) $T_4 = \frac{1}{8 \cdot 2}\left[f(0) + 2f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 2f\left(\frac{3}{8}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.042743$

(b) $M_4 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + f\left(\frac{7}{16}\right)\right] \approx 0.040850$

(c) $S_4 = \frac{1}{8 \cdot 3}\left[f(0) + 4f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 4f\left(\frac{3}{8}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.041478$

9. $f(x) = \dfrac{\ln x}{1+x}$, $\Delta x = \dfrac{2-1}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 2f(1.9) + f(2)] \approx 0.146879$

(b) $M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.147391$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7)$
$+ 2f(1.8) + 4f(1.9) + f(2)]$
≈ 0.147219

10. $f(t) = \dfrac{1}{1+t^2+t^4}$, $\Delta t = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + f(3)\right] \approx 0.895122$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx 0.895478$

(c) $S_6 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + f(3)\right] \approx 0.898014$

11. $f(t) = \sin(e^{t/2})$, $\Delta t = \dfrac{\frac{1}{2}-0}{8} = \dfrac{1}{16}$

(a) $T_8 = \frac{1}{16 \cdot 2}\left[f(0) + 2f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 2f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451948$

(b) $M_8 = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + f\left(\frac{5}{32}\right) + \cdots + f\left(\frac{13}{32}\right) + f\left(\frac{15}{32}\right)\right] \approx 0.451991$

(c) $S_8 = \frac{1}{16 \cdot 3}\left[f(0) + 4f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 4f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451976$

12. $f(x) = \sqrt{1+\sqrt{x}}$, $\Delta x = \dfrac{4-0}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + \cdots + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx 6.042985$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + \cdots + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 6.084778$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 6.061678$

13. $f(t) = e^{\sqrt{t}} \sin t$, $\Delta t = \dfrac{4-0}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx 4.513618$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 4.748256$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 4.675111$

14. $f(x) = \cos\sqrt{x}$, $\Delta x = \frac{4-0}{10} = \frac{2}{5} = 0.4$

(a) $T_{10} = \frac{2}{5 \cdot 2}\left[f(0) + 2f(0.4) + 2f(0.8) + \cdots + 2f(3.2) + 2f(3.6) + f(4)\right] \approx 0.808532$

(b) $M_{10} = \frac{2}{5}[f(0.2) + f(0.6) + f(1) + \cdots + f(3.4) + f(3.8)] \approx 0.803078$